Proceedings of the American Academy of Arts and Sciences.

Vol. XXXVIII. No. 24. — May, 1903.

CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD COLLEGE.

ON FAMILIES OF CURVES WHICH ARE THE LINES OF CERTAIN PLANE VECTORS EITHER SOLENOIDAL OR LAMELLAR.

By B. O. Peirce.

ON FAMILIES OF CURVES WHICH ARE THE LINES OF CERTAIN PLANE VECTORS EITHER SOLENOIDAL OR LAMELLAR.

By B. O. Peirce.

Presented April 8, 1903. Received April 29, 1903.

If a vector function has no component parallel to the axis of z and if the tensors of its components taken parallel to the axes of x and y can be expressed by the scalar point functions $X = \phi_1(x, y)$, $Y = \phi_2(x, y)$, which are independent of z, every line of the vector is a curve parallel to the xy plane, defined by the equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{0}$, and it is sometimes convenient to call the vector itself "plane" and to say that it is "coplanar with" $z = 0$. The projection on the xy plane of any line of such a vector is itself a line of the vector, and a survey of the whole field can be obtained by studying the lines which lie in this plane.

The "divergence" of a vector coplanar with the xy plane is the quantity $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$, and the "curl" of the vector is a vector, directed parallel to the z axis, of intensity $\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$. If the divergence is zero in any region, the vector is said to be "solenoidal" in that region; a vector the curl of which vanishes is said to be "lamellar."

Given any family of curves in the xy plane represented by the equation $u \equiv f_1(x, y) = c_1$, it is possible to find an infinite number of plane vectors which have the u curves as lines, by assuming in each case X at pleasure, and then making

$$Y = -X \cdot \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

The vector (X_0, Y_0) and the vector $(R \cdot X_0, R \cdot Y_0)$, where R is any function of xy, evidently have the same lines, and, if (X_0, Y_0) has for lines the u curves, no other vector has the same lines unless it is of the

form $(R \cdot X_0, R \cdot Y_0)$. Of all the vectors which have the u curves for lines some are lamellar, for, if v is any function orthogonal to u, defined by the equation

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0,$$

so that the curves of the families $u = c_1$, $v = c_2$ cut one another at right angles, the vector which has the components $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$ has for its lines the u curves, and it is lamellar, since

$$\frac{\partial^2 v}{\partial x \cdot \partial y} = \frac{\partial^2 v}{\partial y \cdot \partial x}.$$

If (X_0, Y_0) which has the u curves for lines is lamellar, so is the vector $[X_0 \cdot F(v), Y_0 \cdot F(v)]$, where F represents any ordinary function; and no lamellar vector has the same lines unless it is of the form just given.

If (X_1, Y_1) is a solenoidal vector which has the u curves for its lines, the vector $[X_1 \cdot F(u), Y_1 \cdot F(u)]$ has the same lines and is also solenoidal; no solenoidal vector has these lines unless it can be written in this form. It will soon appear that of all the vectors the lines of which are the u curves, some are always solenoidal, but no vector which has these curves for lines can be both solenoidal and lamellar, unless u happens to satisfy Lamé's condition for isothermal parameters,* that is, unless $\frac{\nabla^2(u)}{h_u^2}$ is expressible as a function of u alone, where

$$\nabla^2(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \; h_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

If a set of orthogonal curvilinear coördinates in the xy plane be defined by the functions

$$u = f_1(x, y), \quad v = f_2(x, y);$$

and if

$$U = \xi(x, y), \quad V = \eta(x, y)$$

represent the magnitudes, at the point (x, y), of the components, taken in the directions in which u and v increase most rapidly, of a vector, Q, coplanar with $z = 0$; it is not difficult to prove, by direct transforma-

* Lamé, Leçons sur les coordonnées curvilignes, p. 31; Leçons sur les fonctions inverses, p. 5; Somoff-Ziwet, Theoretische Mechanik, I. 113 and 128.

tion or otherwise, that the divergence of Q is given by the well known expression

$$\text{Div. } Q \equiv \left(\frac{U}{h_u}\right)\nabla^2(u) + h_u^2 \cdot \frac{\partial}{\partial u}\left(\frac{U}{h_u}\right) + \left(\frac{V}{h_v}\right)\nabla^2(v) + h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right) \quad (1)$$

and that

$$\text{Tensor curl } Q \equiv h_u \cdot h_v \left[\frac{\partial}{\partial u}\left(\frac{V}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{U}{h_u}\right)\right]. \quad (2)$$

If the lines of Q coincide with the u curves, the vector has no component perpendicular to these curves and U is everywhere equal to zero, so that

$$\text{Div. } Q \equiv \frac{V}{h_v} \cdot \nabla^2(v) + h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right), \quad (3)$$

$$\text{Tensor curl } Q \equiv h_u \cdot h_v \cdot \frac{\partial}{\partial u}\left(\frac{V}{h_v}\right), \quad (4)$$

where h_v is the gradient of v.

In applying these expressions it is convenient to remember that

$$\frac{\nabla^2(u)}{h_u^2} = \frac{\partial \log\left(\frac{h_u}{h_v}\right)}{\partial u}, \qquad \frac{\nabla^2(v)}{h_v^2} = \frac{\partial \log\left(\frac{h_v}{h_u}\right)}{\partial v}.$$

It is easy to see from (3) and (4) that the statements which follow are true:

(*a*) If V is to be solenoidal,* we must have

$$\frac{\partial}{\partial v}\left(\log \frac{V}{h_v}\right) = -\frac{\nabla^2(v)}{h_v^2}. \quad (5)$$

The second member of this equation is expressible as a function of u and v; if it be integrated with respect to v while u is considered constant, and if the arbitrary function $\chi(u)$ be added to the result, we shall get $\psi(u, v) + \chi(u)$ the partial derivative of which with respect to v is

$$-\frac{\nabla^2(v)}{h_v^2}; \text{ then } V \equiv h_v \cdot e^{\chi(u)} \cdot e^{\psi(u,v)}. \quad (6)$$

* See equation (18).

(*b*) If V is to be lamellar, we may write

$$V \equiv h_v \cdot \tau\,(v), \tag{7}$$

where τ is any ordinary function. Its divergence is

$$\tau\,(v) \cdot \nabla^2(v) + h_v^2 \cdot \tau'\,(v).$$

If V is to be solenoidal as well as lamellar, we may obtain Lamé's condition immediately by substituting the value of V from (7) in (5).

(*c*) If, like the vector which defines the field of electromagnetic force within an infinitely long cylinder of revolution which carries lengthwise a uniformly distributed, steady current of electricity, V is solenoidal and a function of u only, we must have

$$V\left[\frac{\nabla^2(v)}{h_v} + h_v^2 \frac{\partial}{\partial v}\left(\frac{1}{h_v}\right)\right] = 0,$$

or

$$2 \cdot \nabla^2(v) = \frac{\partial\,(h_v^2)}{\partial v} \tag{8}$$

(*d*) If, like the attraction within a homogeneous, infinitely long, cylinder of revolution, V is lamellar and a function of v only, the gradient of v cannot involve u, so that

$$h_v = f\,(v), \text{ where } f \text{ is arbitrary, or } \frac{\partial h_v}{\partial u} = 0. \tag{9}$$

(*e*) If V is lamellar and a function of u only, $\dfrac{V}{h_v}$ must be independent of u and

$$\frac{\dfrac{\partial h_v}{\partial u}}{h_v} \text{ is a function of } u \text{ only.} \tag{10}$$

In this case h_v is either a function of u only or is expressible as the product of a function of u and a function of v.

(*f*) If V is to be solenoidal* and a function of v only, the expression

$$\frac{\dfrac{\partial h_v}{\partial v}}{h_v} - \frac{\nabla^2(v)}{h_v^2} = \frac{\dfrac{dV}{dv}}{V} \tag{11}$$

* See equation (27).

must be either constant or expressible in terms of v. If V is not lamellar, h_v must in this case involve u.

(g) If V is lamellar and if Ω is a scalar potential function of V, Ω must be expressible in terms of v and the divergence of V is equal to

$$\frac{d\Omega}{dv} \cdot \nabla^2(v) + \frac{d^2\Omega}{dv^2} \cdot h_v^2. \tag{12}$$

(h) If the tensor of V has the same value for all values of x and y, V is lamellar if, and only if, h_v is constant or expressible in terms of v; it is solenoidal if, and only if,

$$2\nabla^2(v) = \frac{\partial h_v^2}{\partial v}. \tag{8}$$

(i) Whatever u is, the vector which has the components

$$X_1 = \frac{f(u)}{h_v} \cdot \frac{\partial v}{\partial x}, \qquad Y_1 = \frac{f(u)}{h_v} \cdot \frac{\partial v}{\partial y},$$

and the vector which has the components

$$X_2 = \frac{\phi(v)}{h_v} \cdot \frac{\partial v}{\partial x}, \qquad Y_2 = \frac{\phi(v)}{h_v} \cdot \frac{\partial v}{\partial y},$$

have the u curves for lines. The tensor of the first is a function of u only, that of the second a function of v only.

(j) If a solenoidal vector has the u lines for curves, its curl must be of the form $\phi(u) \cdot \nabla^2(u) + \phi'(u) \cdot h_u^2$, where ϕ is arbitrary. If, for instance, the u curves are concentric circumferences, the curl of the vector must be expressible as a function of the distance from the centre.

(k) If the tensor of a vector V which has the u curves for lines is a function of u only, its divergence is of the form $V\left(\frac{\nabla^2(v)}{h_v} - \frac{\partial h_v}{\partial v}\right)$. If the u curves are concentric circumferences, V must be solenoidal.

(l) If the tensor of V is expressible in terms of v, the tensor of its curl is $-V\frac{h_u}{h_v} \cdot \frac{\partial h_v}{\partial u}$. If the u curves are straight lines emanating from a point, the curl is zero and the divergence a function of the distance from the point. The velocity in the case of a steady squirt * motion of a gas illustrates this.

* Minchin, Uniplanar Kinematics, 178, Examples 21 and 22.

The Gradients of Functions of Two Independent Variables.

Before we consider briefly some of the equations of condition which have just been stated, it will be well to make a few simple statements concerning the gradients * of functions of x and y.

The gradient of a function may or may not be expressible in terms of the function itself. The gradients of the expressions $(x^2 + y^2)$, $(x^2 - y^2)$ illustrate these two cases.

If the gradient of a function v is equal to $f(v)$, it is possible to form a function of v, $a\int\frac{dv}{f(v)}$, the gradient of which is constant.

If the gradient of a function v is equal to the constant a, it is possible to form two functions of v, namely $\frac{bv}{a}$ and $\frac{1}{a}\int f(v)\,dv$, the gradients of which are equal, respectively, to the arbitrarily chosen constant b and to the arbitrary function $f(v)$.

If the gradient of a function v is either constant or expressible in terms of v, the gradient of any differentiable function of v is expressible as a function of v.

If h_v is neither constant nor expressible in terms of v, no function of v exists the gradient of which is expressible in terms of v.

Since the gradients of two conjugate functions are numerically equal, it is clear that if h_v is expressible in terms of v, not all other functions the gradients of which are functions of v, are themselves expressible in terms of v.

If, for x and y in the expression

$$h_v^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2,$$

the quantities $\lambda = G(x, y)$, $\mu = H(x, y)$ be substituted, we shall obtain the new expression

$$h_v^2 = h_\lambda^2\left(\frac{\partial u}{\partial \lambda}\right)^2 + h_\mu^2\left(\frac{\partial u}{\partial \mu}\right)^2 + 2\left(\frac{\partial \lambda}{\partial x}\cdot\frac{\partial \mu}{\partial x} + \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \mu}{\partial y}\right)\left(\frac{\partial v}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu}\right) \tag{13}$$

and if we write $\lambda = (x + y\,i)$, $\mu = (x - y\,i)$,

* Lamé, Leçons sur les coordonnées curvilignes, p. 6; Maxwell, Treatise on Electricity and Magnetism, § 17.

$$h_v^2 = 4\,\frac{\partial v}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu}: \tag{14}$$

from this last equation it is evident that if the gradient of v vanishes, v is either a function of $x + y\,i$ or a function of $x - y\,i$.

It is often convenient in dealing with differential equations which involve the gradients of functions, to use the independent variables of equation (14) and we may note that u and v, two functions of λ and μ, are conjugate if, and only if,

$$\frac{\partial u}{\partial \lambda} = i\,\frac{\partial v}{\partial \lambda}, \qquad \frac{\partial u}{\partial \mu} = -\,i\,\frac{\partial v}{\partial \mu}. \tag{15}$$

If u and v are orthogonal functions,

$$\frac{\partial u}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu} + \frac{\partial u}{\partial \mu}\cdot\frac{\partial v}{\partial \lambda} = 0. \tag{16}$$

If the gradients of u and v, two real functions of x and y, are everywhere equal while the directions of their gradient vectors are different,

$$\frac{\partial(u-v)}{\partial x}\cdot\frac{\partial(u+v)}{\partial x} + \frac{\partial(u-v)}{\partial y}\cdot\frac{\partial(u+v)}{\partial y} = 0 \tag{17}$$

and the functions $(u - v)$ and $(u + v)$ are orthogonal. The converse of this statement is true. If two orthogonal functions have equal gradients these functions are conjugate.

If the gradient vectors of two functions have the same direction at every point of the $x\,y$ plane, one of these functions is expressible in terms of the other.

The quantities $u = \cos(b\,x - y)$, $v = \sin(b\,y + x)$ illustrate the fact that the gradient of each of two orthogonal functions may be expressible in terms of the function itself.

The quantities $u = x^2 + y^2$, $v = \tan^{-1}\left(\frac{y}{x}\right)$ illustrate the fact that the gradients of both of two orthogonal functions may be expressible in terms of one of the functions.

If the gradient of v, one of two orthogonal functions (u, v) is expressible in terms of u, or is constant, no other but a linear function of v has a gradient expressible in terms of u.

If the gradient of each of two orthogonal functions (u, v) is expressible as a product of a function of u and a function of v, so that

$$h_u = f(u) \cdot F(v), \quad h_v = \phi(u), \quad \psi(v),$$

it is possible to find two functions, $\int \frac{du}{f(u)}$, $\int \frac{dv}{\psi(v)}$, of u and v respectively, the gradient of each of which is expressible in terms of the other.

A solution of Laplace's Equation and any function of its conjugate are orthogonal functions the ratio of the gradients of which is a function of the second function.

Vector Potential Functions of Plane Solenoidal Vectors.

If u, v define a system of orthogonal curvilinear coördinates in the xy plane, and if Q_u, Q_v, Q_z are the components of a vector Q, taken in the directions in which u, v, z increase most rapidly, the components of the curl of Q in these directions are

$$h_v\left[\frac{\partial Q_z}{\partial v} - \frac{\partial}{\partial z}\left(\frac{Q_v}{h_v}\right)\right], \qquad h_u\left[\frac{\partial}{\partial z}\left(\frac{Q_u}{h_u}\right) - \frac{\partial Q_z}{\partial u}\right],$$

$$h_u h_v\left[\frac{\partial}{\partial u}\left(\frac{Q_v}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{Q_u}{h_u}\right)\right].$$

We may denote these quantities by K_u, K_v, K_z, respectively.

If Q is to be a vector potential function of a given solenoidal plane vector $(0, V, 0)$, which has the u curves for lines, we may assume that the components of Q involve u and v only, and since in this case, $K_u = 0$, $K_v = V$, write $Q_z = F(u)$, where $V = -h_u \cdot \frac{dF(u)}{du}$. Any vector of the form $[Q_u, Q_v, F(u)]$, where Q_u, Q_v, are any functions of u and v subject only to the condition $\frac{\partial}{\partial v}\left(\frac{Q_u}{h_u}\right) = \frac{\partial}{\partial u}\left(\frac{Q_v}{h_v}\right)$, is a vector potential function of a solenoidal vector which has the u curves as lines, and there is no vector of this latter kind which does not have as a vector potential a vector of the form just given. In most cases it is simplest to make $Q_u = Q_v = 0$.

If, now, we ask what condition must be satisfied by the function u in order that the curves of the family $u = c$ may be the lines of a vector the tensor of which involves u only, we learn that, since V is of the form $-h_u \cdot F'(u)$, it is necessary and sufficient that h_u be a function of u only. That is

$$\frac{\partial h_u}{\partial v} = 0 \tag{18}$$

Since the divergence of any vector V which has the u curves for lines may be written in the form $h_u \cdot h_v \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_u}\right)$ as well as in the form (3), the condition stated in equation (18) is at once obtained.

If we denote the quantities

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \cdot \partial y}, \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \cdot \partial y}, \frac{\partial^2 v}{\partial y^2},$$

by $p, q, r, s, t, p', q', r', s', t'$, respectively, we have, since u and v are orthogonal,

$$pp' + qq' = 0; \tag{19}$$

whence by differentiation we get

$$p'r + pr' + q's + qs' = 0, \tag{20}$$

$$p's + ps' + q't + qt' = 0. \tag{21}$$

We have, moreover, $\frac{\partial x}{\partial u} = \frac{h_x}{h_u} \cdot \cos(x, u) = \frac{1}{h_u} \cdot \frac{p}{h_u} = \frac{p}{h_u^2}$,

and, similarly, $\frac{\partial y}{\partial u} = \frac{q}{h_u^2}, \quad \frac{\partial x}{\partial v} = \frac{p'}{h_v^2}, \quad \frac{\partial y}{\partial v} = \frac{q'}{h_v^2}, \tag{22}$

so that $\frac{\partial h_u}{\partial u} = \frac{\partial h_u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{1}{h_u^2}\left(\frac{\partial h_u}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial u}{\partial y}\right)$

$$= \frac{p^2 r + 2pqs + q^2 t}{h_u^3}, \tag{23}$$

and $\frac{\partial h_u}{\partial v} = \frac{\partial h_u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{1}{h_v^2}\left(\frac{\partial h_u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial h_u}{\partial y} \cdot \frac{dv}{\partial y}\right)$

$$= \frac{p'[pq(r-t) + s(q^2 - p^2)]}{q h_u h_v^2}.$$

Since, however, $h_u^2 \equiv p^2 + q^2, \quad h_v^2 \equiv p'^2 + q'^2; \quad q^2 \cdot h_v^2 = p'^2 \cdot h_u^2,$

and $\frac{\partial h_u}{\partial v} = \pm \frac{(pqr + (q^2 - p^2)s - pqt)}{h_v \cdot h_u^2}.$ (24)

Equation (18) is equivalent, therefore, to the equation

$$pqr + (q^2 - p^2)s - pqt = 0. \tag{25}$$

If for r and t in (25) we substitute their values as obtained from (20) and (21), we shall get the equation

$$q'^2 r' - 2 p' q' s' + p'^2 t' = 0. \tag{26}$$

and this is (8) in expanded form.

If equation (18) or its equivalent (26) is satisfied, it is evident that by choosing $F(u)$ at pleasure we may find an infinite number of solenoidal vectors which have the u curves as lines and have tensors which involve u only.

A comparison of equations (9) and (18) shows that the condition that the u curves be possible lines of a set of solenoidal vectors the tensors of which involve u only, is the condition that the v curves be possible lines of a set of lamellar vectors the tensors of which involve u only.

If Q is a vector potential function of a solenoidal vector which has the u curves for lines, and a tensor expressible in terms of v, $-h_u F'(u)$ is a function of v, and h_u must be expressible as the product of a function of u and a function of v, that is,

$$h_u = f(u) \cdot \phi(v). \tag{27}$$

If for u in this differential equation we substitute w, defined by the equation $w \equiv \int \frac{du}{f(u)}$, we get the simpler equation

$$h_w = \phi(v) \quad \text{or} \quad \frac{\partial h_w}{\partial w} = 0. \tag{28}$$

It is to be noticed that w has the same lines as u, and that (27) and (28) define the same curves; the equations (11) and (28) are evidently equivalent.

If u is such that a solenoidal vector, V, can be found which has the u curves for lines and a tensor expressible in terms of v, its x and y components are $\left(\frac{V}{h_v} \cdot \frac{\partial v}{\partial x}, \frac{V}{h_v} \cdot \frac{\partial v}{\partial y}\right)$. If we denote these components by X, Y, every other solenoidal vector which has the same lines has components of the form $X \cdot \psi(u)$, $Y \cdot \psi(u)$, and the vector is not a function of v alone unless the factor $\psi(u)$ degenerates into a constant, and the vector is a simple multiple of V.

A comparison of (10) and (27) shows that if the u curves are possible lines of a solenoidal vector the tensor of which is expressible in terms of

v, the v curves are the possible lines of a lamellar vector the tensor of which is a function of v only.

If $\frac{\partial h_u}{\partial v} = 0$ and $\frac{\partial h_v}{\partial u} = 0$, the u curves are the lines of a set of solenoidal vectors the curls of which are expressible in terms of u only; and the v curves are the lines of a set of solenoidal vectors the curls of which are expressible in terms of v only.

Possible Systems of Isothermal Straight Lines and Isothermal Circles in a Plane.

(1) Let $\alpha x + \beta y = 1$, where α and β are any functions of a single parameter u, represent a family of straight lines in the xy plane, then we may write

$$\frac{\partial u}{\partial x} = \frac{-\alpha}{\alpha' x + \beta' y}, \qquad \frac{\partial u}{\partial y} = \frac{-\beta}{\alpha' x + \beta' y}, \qquad h_u^2 = \frac{\alpha^2 + \beta^2}{(\alpha' x + \beta' y)^2},$$

$$\frac{\nabla^2(u)}{h_u^2} = \frac{2(\alpha\alpha' + \beta\beta')}{\alpha^2 + \beta^2} - \frac{\alpha'' x + \beta'' y}{\alpha' x + \beta' y}. \tag{29}$$

If then $\frac{\nabla^2(u)}{h_u^2}$ is to be a function of u only, the last term in the second member of this last equation must be expressible in terms of u only, and we have $\alpha' = 0$, or $\beta' = 0$, or, in general, $\alpha'' : \alpha' = \beta'' : \beta'$, so that $\alpha = c\beta + d$, where c and d are constants of integration. The equation of the family of lines must be of the form $(c\beta + d)x + \beta y = 1$, and the lines all pass through the fixed point $\left(\frac{1}{d}, -\frac{c}{d}\right)$, which may be chosen at pleasure. If $d = 0$, the lines are parallel.

(2) Let $x^2 + y^2 - 2\alpha x - 2\beta y = \gamma$, where α, β, γ are functions of a single parameter u, represent a family of circumferences in the xy plane, then we may write

$$\frac{\partial u}{\partial x} = \frac{2(x - \alpha)}{2\alpha' x + 2\beta' y + \gamma'}, \qquad \frac{\partial u}{\partial y} = \frac{2(y - \beta)}{2\alpha' x + 2\beta' y + \gamma'},$$

$$h_u^2 = \frac{4(\alpha^2 + \beta^2 + \gamma)}{(2\alpha' x + 2\beta' y + \gamma')^2},$$

$$\frac{\nabla^2(u)}{h_u^2} = \frac{2\alpha\alpha' + 2\beta\beta' + \gamma'}{\alpha^2 + \beta^2 + \gamma} - \frac{1}{\alpha^2 + \beta^2 + \gamma} \cdot \frac{2\alpha'' x + 2\beta'' y + \gamma''}{2\alpha' x + 2\beta' y + \gamma'}. \tag{30}$$

If $\frac{\nabla^2(u)}{h_u^2}$ is a function of u only, the last term in the second member of (30) must be expressible in terms of u only. If $\alpha' = \beta' = 0$, we have a family of concentric circumferences. In general we may write $\alpha'' : \alpha' = \beta'' : \beta' = \gamma'' : \gamma'$, or $\beta = m\alpha + n$, $\gamma = 2k\alpha + l$, so that the equation of the circles must be of the form

$$x^2 + y^2 - 2\alpha x - 2y(m\alpha + n) - 2k\alpha - l = 0, \qquad (31)$$

where α is the only parameter. If we represent the first member of this equation by S_α, the equation $S_{\alpha_1} - S_{\alpha_2} = 0$ represents the straight line through the points of intersection of the circles which correspond to the two values α_1, α_2 of the parameter. In this case the line is $x + my + k = 0$, whatever the values of α_1 and α_2, therefore, as is well known, the system of isothermal circles* must pass through two fixed, real or imaginary, points.

Functions the Gradients of which are expressible in Terms of the Functions themselves.

Several of the conditions stated in the previous pages [see (*d*), (*h*), and equation (18)] require that the gradient of a function be expressible in terms of the function itself, so that the normal derivative of the function has the same numerical value at all points of any one of its curves of level. We may state this requirement in a somewhat simpler form, however, if we remember that since the gradient of any function, ϕ, of u is equal to $\phi'(u) \cdot h_u$, the lines of all functions which satisfy the equation $h_u = f(u)$, whatever f may be, are included in the lines of functions which satisfy the equation $h_u = k$, where k is any constant (for instance 1). Every such family of lines forms a set of parallel curves. We have to solve, then, the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = k^2, \qquad (32)$$

one of the standard forms for partial differential equations of the first order.

Its complete integral is

$$u = \alpha x + y\sqrt{k^2 - \alpha^2} + c,$$

and its general integral,

* Darboux, Leçons sur la théorie générale des surfaces.

$$u = ax + y\sqrt{k^2 - a^2} + \psi(a), \tag{33}$$

subject to the condition

$$0 = x - \frac{ay}{\sqrt{k^2 - a^2}} + \psi'(a).$$

The equation $h_u = k$ is also equivalent to the equation,

$$\frac{\partial u}{\partial \lambda} \cdot \frac{\partial u}{\partial \mu} = 1. \tag{34}$$

The complete integral of (35) is

$$u = a\lambda + \frac{\mu}{a} + c,$$

and its general integral* may be found by eliminating a between the equations,

$$u = a\lambda + \frac{\mu}{a} + \phi(a), \qquad 0 = \lambda - \frac{\mu}{a^2} + \phi'(a). \tag{35}$$

If u is to be harmonic while h_u is expressible in terms of u, u is of the form $\phi(\lambda) + \psi(\mu)$, where $\lambda = x + yi$, $\mu = x - yi$. Since

$$h_u^2 = 4\frac{\partial u}{\partial \lambda} \cdot \frac{\partial u}{\partial \mu},$$

we must have

$$4\,\phi'(\lambda) \cdot \psi'(\mu) = 4f[\phi(\lambda) + \psi(\mu)],] \tag{36}$$

and if we differentiate both sides of this equation with respect to λ and μ we shall get

$$\phi''(\lambda) \cdot \psi'(\mu) = \phi'(\lambda) \cdot f'[\phi(\lambda) + \psi(\mu)],$$

$$\phi'(\lambda) \cdot \psi''(\mu) = \psi'(\mu) \cdot f'[\phi(\lambda) + \psi(\mu)],$$

whence

$$\frac{\phi''(\lambda)}{[\phi'(\lambda)]^2} = \frac{\psi''(\mu)}{[\psi'(\mu)]^2}. \tag{37}$$

Since the first member of (37) involves λ only, and the second member μ only, we may equate each member to a constant, $-k^2$, and consider separately the cases where k is or is not zero.

* Forsyth, Differential Equations, p. 307.

(1) If $k = 0$, $\phi''(\lambda) = 0$, $\psi''(\mu) = 0$, $\phi(\lambda) = c\lambda + m$,

$$\psi(\mu) = d\mu + n, \quad \text{and} \quad u = c\lambda + d\mu + g.$$

If c and $\pm d$ are either real and equal, or conjugate complex quantities, the u curves are a set of real parallel straight lines.

(2) If $k \neq 0$, $\phi'(\lambda) = \frac{1}{k^2\lambda + m}$, $\psi'(\mu) = \frac{1}{k^2\mu + n}$,

$$\phi(\lambda) = \frac{1}{2^2} \cdot \log(k^2\lambda + m) + a, \quad \psi(\mu) = \frac{1}{k^2} \cdot \log(k^2\mu + n) + b,$$

and if the constants of integration are so chosen as to make u real or purely imaginary, the u curves are a set of parallel, that is, concentric, circumferences.

Every family of isothermal lines which are the curves of a function u which satisfies (18) is either a set of parallel straight lines or a set of concentric circumferences. No other families of parallel curves are isothermal.

The Equation $h_u \cdot \frac{\partial h_u}{\partial u} = \nabla^2(u)$.

We have seen that equations (18) and (26) are equivalent; this equation, therefore, defines the families of straight lines which form the orthogonal trajectories of the families of parallel curves defined by the equation $h_v = f(v)$, and we may write

$$q^2 r - 2pqs + p^2 t = 0. \tag{38}$$

Monge's method yields the first integral $u = F\left(\frac{p}{q}\right)$, (39)

and of this equation

$$u = \phi\left(\frac{a + y}{c - x}\right) \tag{40}$$

is the complete integral and

$$u = \phi\left(\frac{a + y}{\psi(a) - x}\right), \tag{41}$$

where $\psi(a) - x = (a + y)\,\psi'(a)$, the general integral.

Every family of straight lines in the xy plane, that is every set of lines defined by the equation $ax + py = 1$, where a and p are arbitrary

functions of a single parameter, are contained as, of course, they should be, in this general integral.

It is evident that every family of isothermal lines which are the curves of a function u which satisfies (38) is a set of straight lines which pass through a point.

Transformation of the Equation $h_u = f(v)$.

Given a function which satisfies (10) or (27), there always exists a function which has the same lines, and a gradient expressible in terms of the orthogonal function alone. The lines of all functions which satisfy these equations are therefore those of functions which satisfy an equation of the form

$$\frac{\partial h_u}{\partial u} = 0 \tag{42}$$

or

$$p^2 r + 2\,p\,q\,s + q^2 t = 0. \tag{43}$$

If we take advantage of the Principle of Duality and make $p = x'$, $q = y'$, $p\,x + q\,y - z = z'$, we shall get the transformed equation

$$x'^2 \cdot t' - 2\,x'y' \cdot s' + y'^2 \cdot r' = 0, \tag{44}$$

and if then we put, $m = -2\log(x^2 + y^2)$, $n = \tan^{-1}\left(\frac{y}{x}\right)$, the result is

$$\frac{\partial u}{\partial m} = \frac{\partial^2 u}{\partial n^2} \tag{45}$$

which is equivalent to Fourier's familiar equation for the linear flow of heat.

If u is to be harmonic, while $\frac{\partial h_u}{\partial u} = 0$, we may write

$$u = \phi(x + yi) + \psi(x - yi),$$

and substitute this value in equation (43).

The resulting equation is

$$[\phi'(x + yi)]^2 \cdot \psi''(x - yi) + [\psi'(x - yi)]^2 \cdot \phi''(x + yi) = 0, \tag{46}$$

or

$$\frac{\psi''(x - yi)}{[\psi'(x - yi)]^2} = -\frac{\phi''(x + yi)}{[\phi'(x + yi)]^2}. \tag{47}$$

This last equation is possible only if each member is constant, $(-k^2)$, whence $\psi = \frac{1}{k^2} \cdot \log(k^2\mu + m) + a$, $\phi = -\frac{1}{k^2} \cdot \log(k^2\lambda + n) + b$, and the u curves are a family of straight lines meeting in some point. No other families of isothermal lines are possible curves of scalar functions, the gradients of which are expressible in terms of the corresponding orthogonal functions.

The Jefferson Laboratory, Cambridge, Mass.,
April, 1903.